SUGGESTED SOLUTION TO HOMEWORK 6

JUNHAO ZHANG

Problem 1. Let $\{x_n\}_{n\geq 1}$ be a sequence in an inner product space. Show that conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ imply $x_n \to x$.

Proof. Since $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$, therefore for arbitrary $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that for $n > N_1$,

$$|\|x_n\| - \|x\|| < \min\left\{\frac{\sqrt{3}\varepsilon}{3}, \frac{\varepsilon^2}{6\|x\|}\right\},$$

and for $n > N_2$,

$$|\langle x_n, x\rangle - \langle x, x\rangle| < \frac{\varepsilon^2}{6},$$

then for $n > \max\{N_1, N_2\}$,

$$\begin{aligned}
\|x_n - x\|^2 =& \langle x_n - x, x_n - x\rangle \\
=& (\|x_n\| - \|x\|)^2 + (2\|x_n\|\|x\| - \langle x_n, x\rangle - \langle x, x_n\rangle) \\
\leq& (\|x_n\| - \|x\|)^2 + 2\|x\| \cdot |\|x_n\| - \|x\|| \\
&+ |\langle x, x\rangle - \langle x_n, x\rangle| + \left|\overline{\langle x, x\rangle - \langle x_n, x\rangle}\right| \\
<& \varepsilon^2,
\end{aligned}$$

which implies that $x_n \to x$. □

Problem 2. Let x and y be linearly independent vectors in an inner product space such that $\|x\| = \|y\| = 1$. Show that

$$\|tx + (1-t)y\| < 1, \quad \text{for } 0 < t < 1.$$

Proof. Since $\|x\| = \|y\| = 1$, then for $0 < t < 1$, by Cauchy inequality,

$$\begin{aligned}
\|tx + (1-t)y\|^2 =& \langle tx + (1-t)y, tx + (1-t)y\rangle \\
\leq& t^2\|x\|^2 + 2t(1-t)\langle x, y\rangle + (1-t)^2\|y\|^2 \\
\leq& t^2\|x\|^2 + 2t(1-t)\|x\|\|y\| + (1-t)^2\|y\|^2 \\
\leq& 1,
\end{aligned}$$

where the equality holds if and only if x and y are linear dependent, therefore for linearly independent vectors x and y,

$$\|tx + (1-t)y\| < 1.$$

□

Problem 3. Prove that in an inner product space, $x \perp y$ if and only if

$$\|x + \lambda y\| = \|x - \lambda y\|,$$

for all scalars $\lambda \in \mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Proof. $\Rightarrow$: Let $x \perp y$, then

$$\langle x, y\rangle = 0,$$

therefore for all $\lambda \in \mathbb{F}$,

$$\begin{aligned}\|x+\lambda y\| =& \sqrt{\langle x+\lambda y, x+\lambda y\rangle}\\ =& \sqrt{\langle x, x\rangle + \langle \lambda y, \lambda y\rangle}\\ =& \sqrt{\langle x-\lambda y, x-\lambda y\rangle}\\ =& \|x-\lambda y\|.\end{aligned}$$

$\Leftarrow$: Suppose for all $\lambda \in \mathbb{F}$,

$$\|x+\lambda y\| = \|x-\lambda y\|,$$

then by Polarization indentities, if $\mathbb{F} = \mathbb{R}$,

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = 0,$$

if $\mathbb{F} = \mathbb{C}$,

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \mathbf{i}\|x+\mathbf{i}y\|^2 - \mathbf{i}\|x-\mathbf{i}y\|^2\right) = 0.$$

Hence for either $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we have $x \perp y$. $\square$

Email address: `jhzhang@math.cuhk.edu.hk`